



Fracture Mechanics

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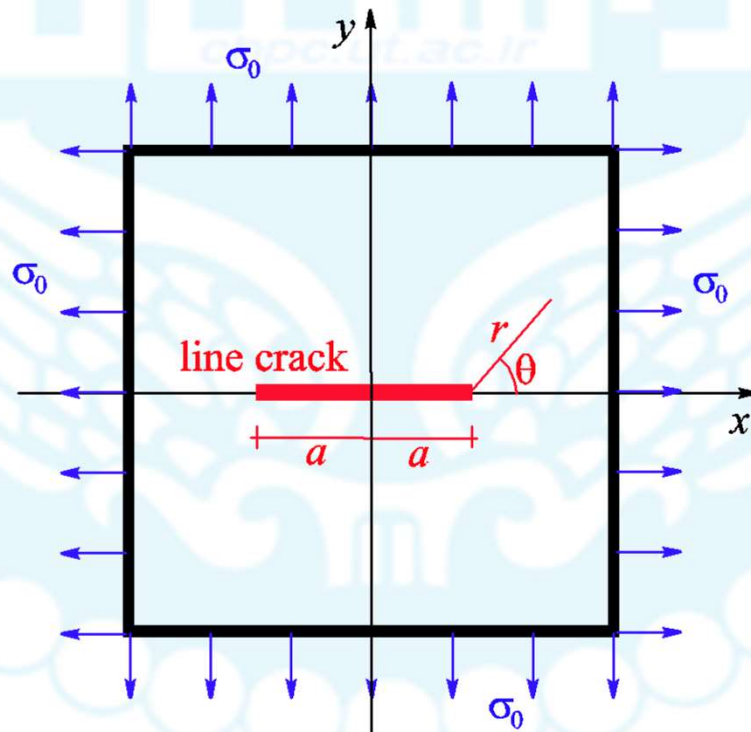


Westergaard Analysis of a Line Crack

Westergaard analysis of a line crack

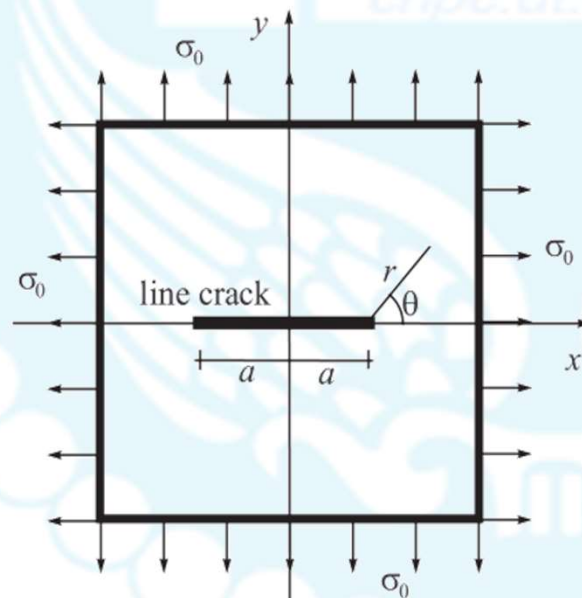
Consider an infinite plate with a central traction-free crack of length $2a$ subjected to uniform biaxial stress σ_0 .

From Inglis solution, we know that there would be a theoretically infinite stress at the tip of the crack, however neither the nature of the singularity nor the stress field can be derived from it.

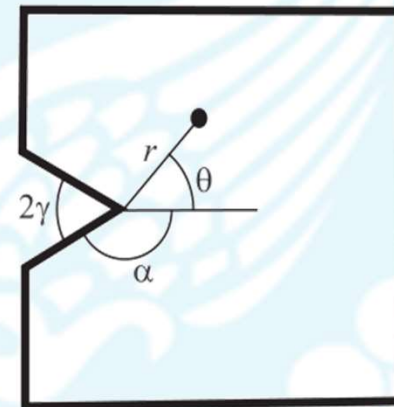


Westergaard analysis of a line crack

There are two classical solutions for the crack problem. The first one was proposed by Westergaard, and the later by Williams. Whereas the first one is simpler to follow, the second has the advantage of being extended to cracks at the interface of two different homogeneous isotropic materials and be applicable for V notches.



a) Westergaard's crack

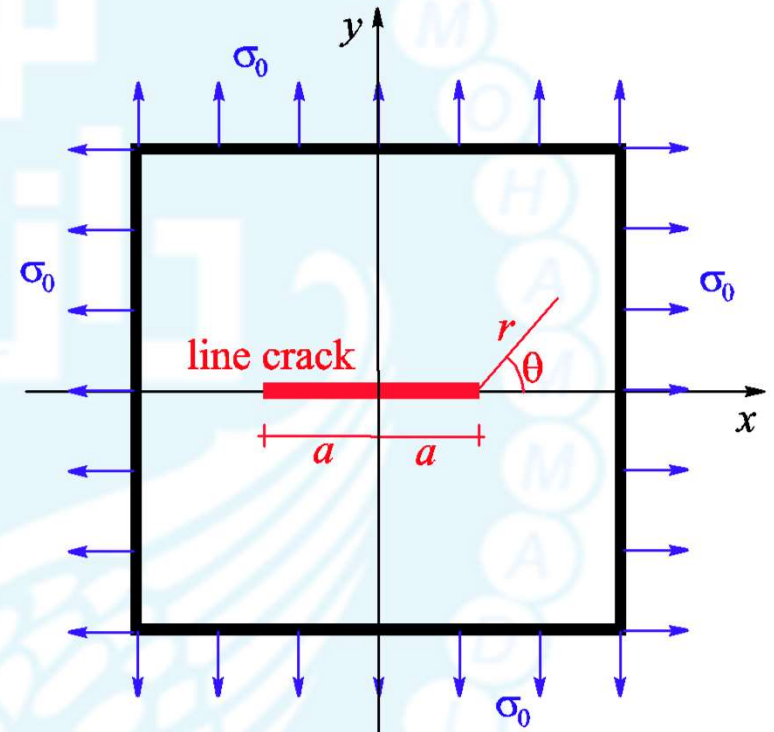


b) William's wedge

Westergaard analysis of a line crack

Westergaard (1939) proposed the following biharmonic stress function Φ as a solution to the crack problem:

$$\Phi = \operatorname{Re} \bar{\bar{\phi}}(z) + y \operatorname{Im} \bar{\phi}(z)$$



where $\bar{\phi}$ and $\bar{\bar{\phi}}$ are the first and second integrals of $\phi(z)$.

Westergaard analysis of a line crack

The stress components then become:

$$\Phi = \operatorname{Re} \bar{\bar{\phi}}(z) + y \operatorname{Im} \bar{\bar{\phi}}(z)$$

$$\sigma_{xx} = \operatorname{Re} \phi(z) - y \operatorname{Im} \phi'(z)$$

$$\sigma_{yy} = \operatorname{Re} \phi(z) + y \operatorname{Im} \phi'(z)$$

$$\sigma_{xy} = -y \operatorname{Re} \phi'(z)$$

where ϕ' is the first derivative of $\phi(z)$.

Westergaard analysis of a line crack

The complex function ϕ is determined so that the boundary conditions are satisfied along the crack and at infinity:

1. Along the crack:

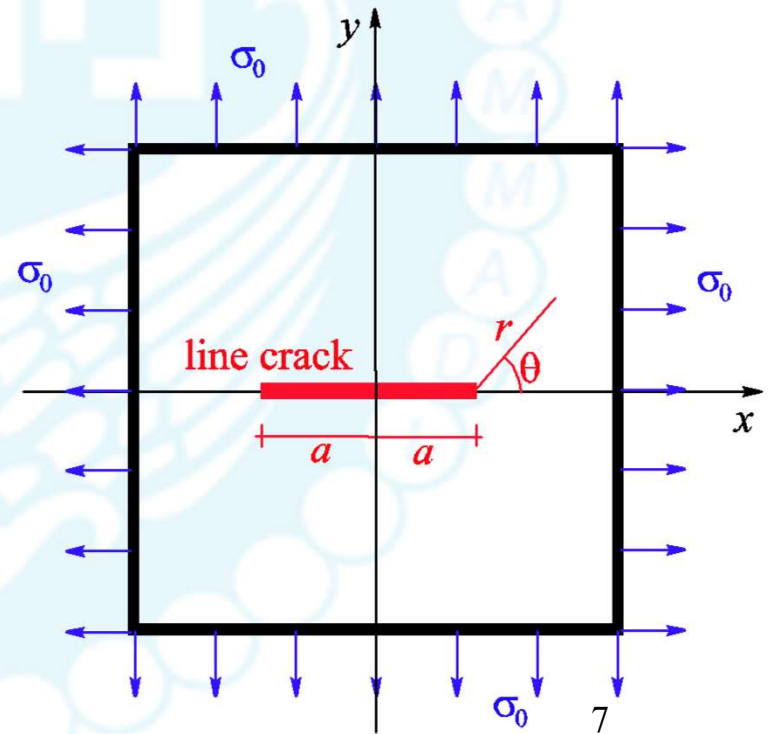
at $y=0$ and $-a < x < a$: $\sigma_{yy}=0$ (traction free crack).

2. At infinity:

at $y= \pm\infty$: $\sigma_{yy}=\sigma_0$

σ_{yy} at $y=0$ reduces to

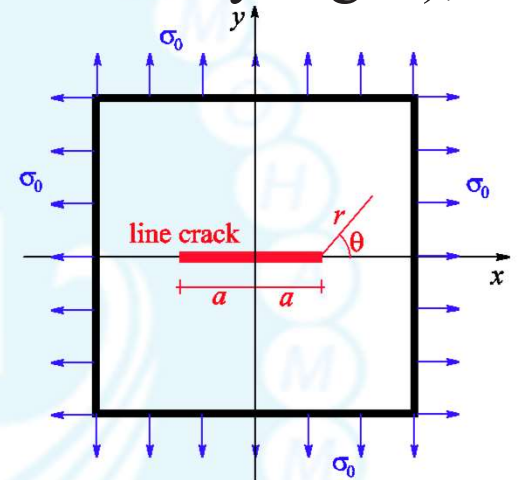
$$\sigma_{yy}\Big|_{y=0} = \text{Re } \phi(z)$$



Westergaard analysis of a line crack

Furthermore, we expect $\sigma_{yy} \rightarrow \sigma_0$ as $x \rightarrow \infty$, and σ_{yy} to be greater than σ_0 when $|x-a| > \varepsilon$ (due to anticipated singularity predicted by Inglis), thus a possible choice for σ_{yy} would be

$$\sigma_{yy} = \frac{\sigma_0}{1 - \frac{a}{x}} \longrightarrow \sigma_{yy} = \frac{\sigma_0}{1 - \frac{a^2}{x^2}}$$



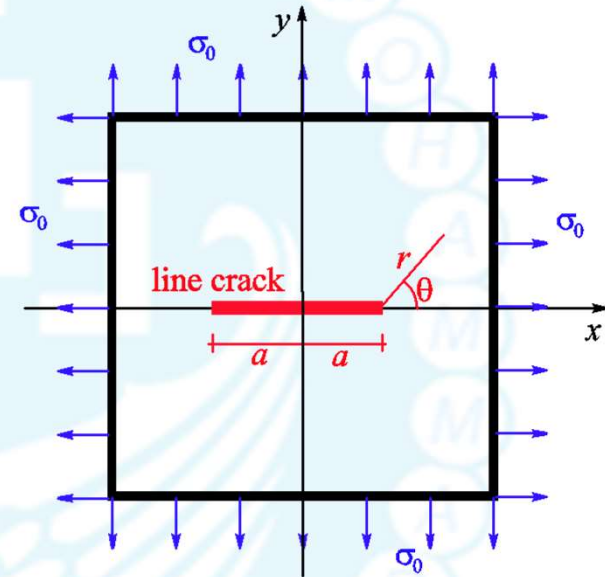
However, we also need to have $\sigma_{yy} = 0$ when $y = 0$ and $-a < x < a$, thus the function $\phi(z)$ should become imaginary along the crack:

$$\sigma_{yy} = \frac{\sigma_0}{1 - \frac{a^2}{x^2}} \longrightarrow \sigma_{yy} = \operatorname{Re} \left(\frac{\sigma_0}{\sqrt{1 - \frac{a^2}{x^2}}} \right)$$

Westergaard analysis of a line crack

Thus, the complex function ϕ that satisfies the boundary conditions along the crack and at infinity can be written as:

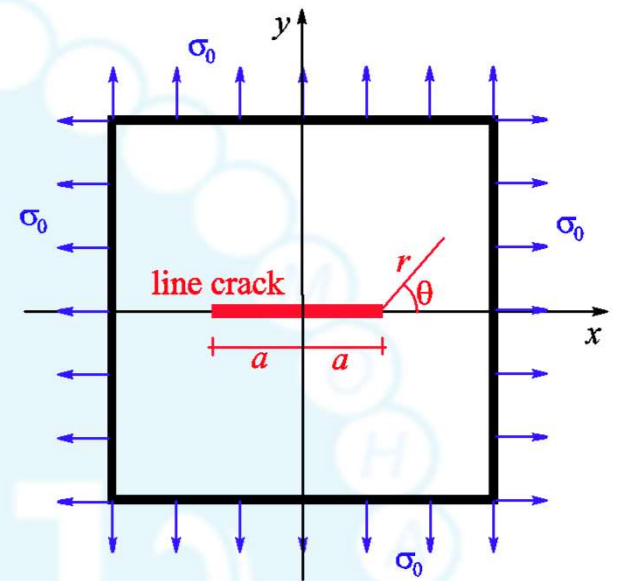
$$\phi(z) = \frac{\sigma_0}{\sqrt{1 - \frac{a^2}{z^2}}}$$



1. Along the crack: $\sigma_{yy}=0$
2. At infinity: $\sigma_{yy}=\sigma_0$

Westergaard analysis of a line crack

$$\phi(z) = \frac{\sigma_0}{\sqrt{1 - \frac{a^2}{z^2}}}$$



Westergaard made simplifications for points near the crack tip:

$$\text{For } r \ll a: \begin{cases} \operatorname{Re} \phi(z) \approx \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \\ y \operatorname{Im} \phi'(z) = \sigma_0 \sqrt{\frac{a}{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \end{cases}$$

Westergaard analysis of a line crack

Then, the final near crack tip solutions are obtained ($r \ll a$) (Meguid, 1989):

$$\sigma_{xx} = \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \dots$$

$$\sigma_{yy} = \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \dots$$

$$\sigma_{xy} = \sigma_0 \sqrt{\frac{a}{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \dots$$

Neglecting the higher order terms is only acceptable for small values of r around the crack tip.

Important notes:

1. The stress components near the crack tip are related to $(r^{-0.5})$.
2. The simplified solutions are not accurate far from the crack tip. It is noted that if $r \rightarrow \infty$, all stress components become null, which is not physical and does not satisfy the far traction boundary conditions.
3. The trigonometric functions are more or less observed in crack tip stress fields in various problems.
4. The stress components are directly related to applied traction σ_0 .
5. The stress components are directly related to the square root of crack length $(a^{0.5})$.
6. Distribution of stress components in radial (r) and angular (θ) directions are independent functions. These functions will later be used to enrich the numerical solutions for fracture mechanics.

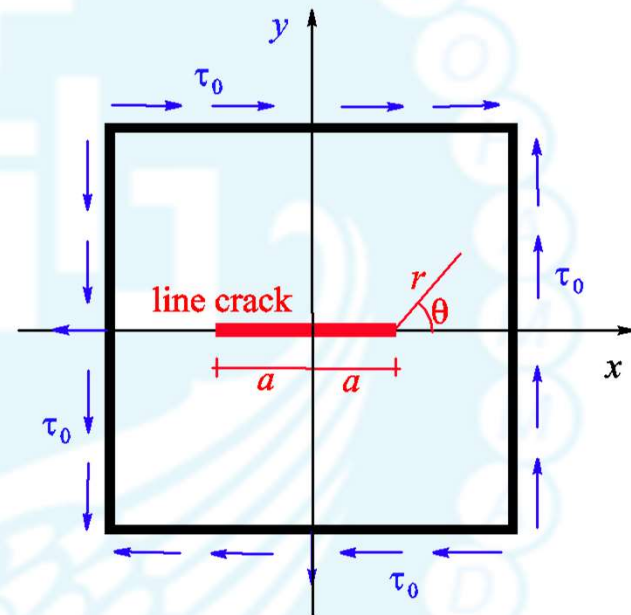
Westergaard analysis of a line crack

Using a similar approach, the stress field around a crack tip in an infinite plate subjected to shear stresses τ_0 can be expressed as:

$$\phi_s(z) = \frac{\tau_0}{\sqrt{1 - \frac{a^2}{z^2}}}$$

and

$$\Phi_s(z) = -y \operatorname{Re} \bar{\phi}_s(z)$$



HW P2-C04#1:

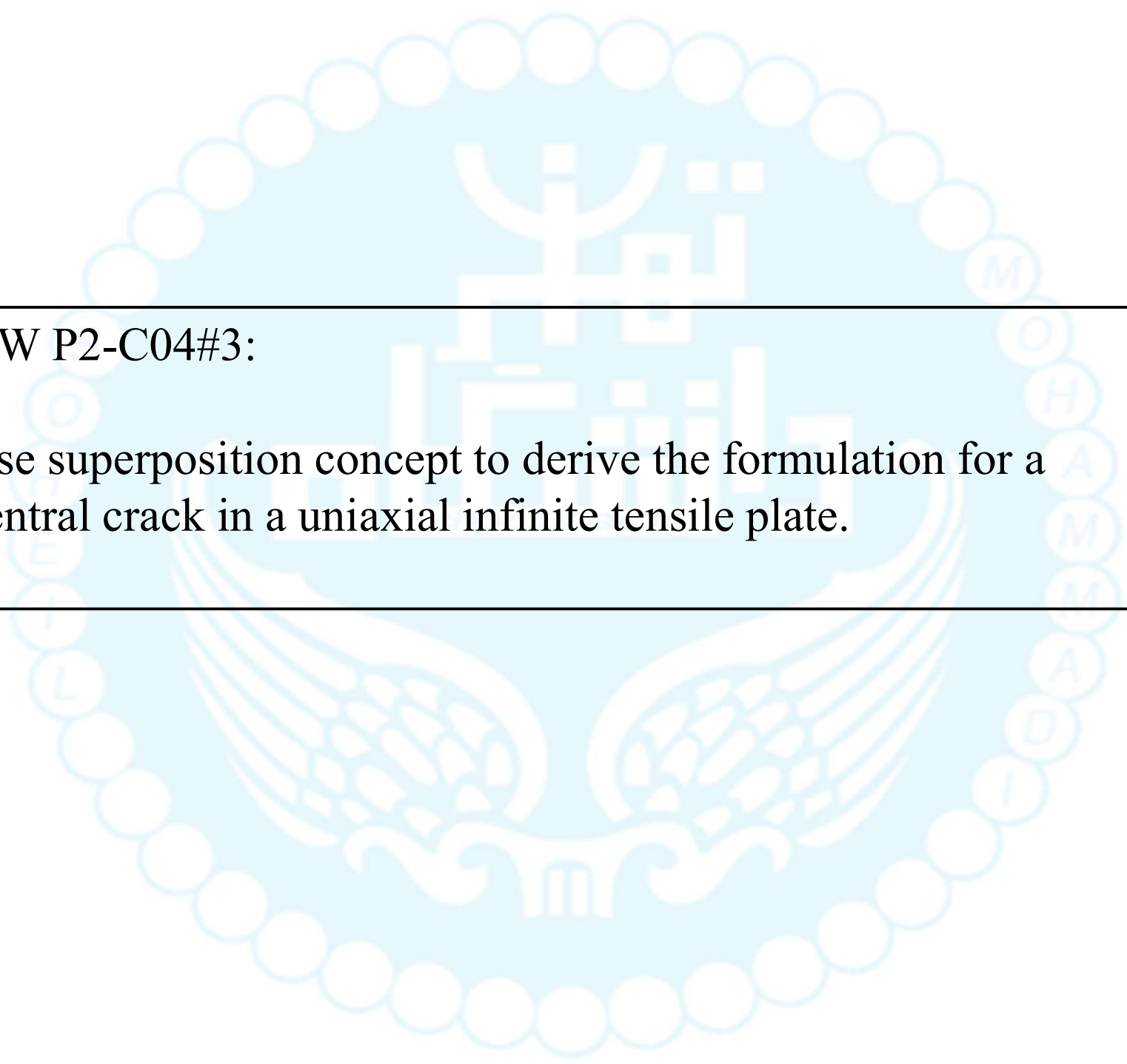
Derive the formulations of stress components based on the exact stress function.

$$\phi(z) = \frac{\sigma_0}{\sqrt{1 - \frac{a^2}{z^2}}}$$

HW P2-C04#2:

Draw the stress contours around a central crack in an infinite tensile plate:

- 1- Based on the simplified formulations
- 2- Based on the original stress function
- 3- Draw the contours of error



HW P2-C04#3:

Use superposition concept to derive the formulation for a central crack in a uniaxial infinite tensile plate.

